Mis-Specification and Frequency Dependence in a New Keynesian Phillips Curve

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Abstract

A Phillips curve is at the center of New Keynesian macroeconomic modeling. Yet this relationship has proven quite difficult to detect in post-1984 quarterly data. Here we provide compelling evidence that previous models quantifying the dynamic relationship between inflation and unemployment rates have been mis-specified in their assumption that the coefficient on unemployment is a constant. Instead, we find that this coefficient is frequency-dependent: the inflation impact of a fluctuation in the unemployment rate differs for a fluctuation which is part of a smooth pattern of changes versus a fluctuation which is an isolated event, just as the Friedman-Phelps "natural rate" hypothesis suggests.

In particular, we analyze the Blanchard/Gali (2005) Phillips Curve regression specification using a newly developed econometric technique capable of consistently estimating the frequency dependence in a feedback relationship. Explicitly allowing for feedback in such a relationship is essential because the two-sided nature of the Fourier transformations used in previous frequency domain studies otherwise confounds the analysis, leading to inconsistent parameter estimates. Using one-sided filtering to allow for observed feedback in the relationship, we find statistically significant frequency dependence. In particular, using quarterly data from 1984:I-2003:IV, we find an economically and statistically significant inverse relationship between inflation and unemployment for higher-frequency unemployment rate fluctuations – with periods less than or equal to about one year – but no evidence for an effect of lower-frequency unemployment rate fluctuations. In contrast, an analogous model ignoring frequency dependence finds no statistically significant relationship at all between inflation and unemployment rates during this sample period.

1 Introduction

Few macroeconomic relationships have received as much attention as the Phillips curve, which postulates an inverse relationship between inflation and the unemployment rate.[1] This relationship is central to contemporary monetary policy: one cannot hope to appropriately conduct such policy without an adequate understanding of short-run inflation dynamics. This relationship is also central to New Keynesian macroeconomics: a Phillips curve relation forms one of the three key equations in most New Keynesian macro models.

Yet despite its importance to macroeconomics, the nature of this relationship remains strongly contested. The very existence of a New-Keynesian Phillips curve has, in fact, been called into question; indeed, this was the subject of sharply pointed exchanges during the September 2005 conference *Quantitative Evidence on Price Determination*, which was jointly sponsored by the Federal Reserve Board and the *Journal of Money, Credit and Banking.* Two major reasons for doubting its existence are: 1), in post-1984 data, it is difficult to detect a New Keynesian Phillips curve – a result confirmed in this study; and 2), there is reason to doubt that expected inflation has an important role in causing current inflation (Rudd and Whelan, 2003). Below, we argue that these both fundamentally stem from an important empirical mis-specification: the relationship between inflation and the unemployment rate is implicitly assumed to be constant across frequencies, an assumption which we find is not consistent with the data. Our finding of frequency-dependence in this relationship is both statistically and economically significant. Thus, empirical specifications which do not take this into account will yield a distorted picture of the true relationships. In fact, the New Keynesian Phillips curve is strongly supported by the data once this frequency dependence is treated appropriately.

What lies behind this frequency dependence? One interpretation is the existence of a natural rate of unemployment. If only deviations of the unemployment rate from its current natural rate influence inflation, then empirical approaches which ignore the natural rate, or treat it inappropriately, will likely fail to uncover the true relationship between inflation and its determinants.[2]

[1] Although credit for the discovery of this relationship generally goes to Phillips (1958), one could argue that the original discovery was due to Fisher (1926).

[2] Estrella and Mishkin (1999) argue that it is important to distinguish between a natural rate of unemployment and a NAIRU. To the extent that this distinction is important, one could argue that our empirical

However, almost all current approaches to the natural rate of unemployment impose arbitrary, and most likely counterfactual, assumptions about its dynamic evolution, as discussed in Ashley and Verbrugge (2006a). A more satisfactory approach is to impose fewer assumptions about natural rate dynamics, and let the data speak more fully about its evolution.

Below we review the approach introduced in Ashley and Verbrugge (2006a) for detecting and modeling frequency dependence in an estimated regression model coefficient; we then apply this approach to a recent (Blanchard and Gali, 2005) New Keynesian "hybrid" Phillips curve relationship.

The search for frequency-dependence in the Phillips curve is motivated by the fact that the Friedman-Phelps natural rate hypothesis strongly suggests that the relationship between the inflation rate and the unemployment rate is actually *frequency-dependent*; that is, the relationship between low-frequency movements in the inflation rate (corresponding to the prevailing steady-state inflation rate) and low frequency movements in the unemployment rate (corresponding to changes in the natural rate[3]) will likely be quite different from the relationship of higher-frequency movements in the inflation rate to higher-frequency movements in the unemployment rate. In essence, the Friedman-Phelps formulation suggests that the high frequency movements in these two time series may well have the inverse relationship suggested by Phillips, while the low frequency movements will be unrelated. Clearly, if such frequency-dependence is empirically significant, then a standard Phillips curve model which assumes that the same relationship obtains at all frequencies will yield coefficient estimates that consistently characterize neither of these two distinct relationships, and researchers may well draw erroneous conclusions.

As we discuss below (and in more detail in Ashley and Verbrugge 2006a), all presently-available methods for detecting and modeling frequency dependence fail to provide consistent parameter estimates when feedback is present in the relationship, as is the case in the inflation-unemployment relationship. This failure is due to the two-sided nature of the filtering – Hodrick-Prescott, Baxter-King, or even ordinary X-11 seasonal adjustment – used in these approaches to isolate a specific

work is actually uncovering a NAIRU rather than a natural rate. See the Winter 1997 issue of the *Journal of Economic Perspectives*.

[3]Hall (1999) and Cogley and Sargent (2001) argue that the low frequency trend component of the unemployment rate is an estimate of the natural rate; Staiger, Stock and Watson (2001) adopt this argument.

range of frequencies for analysis. Fundamentally, the two-sided filtering interacts with the feedback in the relationship to induce correlations between the filtered series and the relevant regression error terms, thus producing inconsistent parameter estimates. This suggests caution in interpreting the coefficients from *any* dynamic regression using filtered data which may be part of a feedback relationship.

The approach used here is formulated in the time domain, so it is easy to implement using ordinary regression software. Moreover, it does not require any *a priori* imposition of assumptions regarding the relevant frequency ranges; since the new procedure does not require any specification of the dynamics of the natural rate of unemployment, its validity does not hinge on the correctness of such a specification.

Applying this new technique to allow for both frequency dependence and feedback in the Blanchard/Gali New Keynesian Phillips curve, we find statistically significant frequency dependence in the Phillips curve relationship, of a sort that is consistent with the Friedman-Phelps theory. In particular, the data indicate that there is a statistically significant inverse relationship betwen inflation and unemployment – but this significant relationship is restricted to medium- and high-frequency unemployment rate fluctuations, i.e. fluctuations with periods less than about one year. Our results support the existence of the Blanchard/Gali hybrid NKPC, once frequency dependence is taken into consideration.

The outline of the remainder of the paper is as follows. Section 2 presents the underlying macroeconomic theory and briefly discusses prior empirical work. Section 3 describes the econometric methodology proposed here, and in particular includes a critique of two-sided filtering in the presence of feedback. Section 4 presents the empirical results on the Phillips curve. Section 6 concludes the paper.

2 Theory and Prior Empirical Work

The "New Keynesian Phillips curve" (NKPC) is at the heart of the standard New Keynesian model; in its simplest form, the NKPC is

$$\pi_t = \beta E \pi_{t+1} + \kappa \left(y_t - y_t^*\right) \tag{1}$$

where π is inflation, y is (log) natural output, and $(y - y^*)$ is the output gap. This version of the NKPC is not an empirical success, however, as a number of authors have noted (see, e.g., Rudd and Whelan 2003). Two of the major empirical criticisms are: first, this specification is inconsistent with the high level of persistence in inflation (in (1), inflation persistence has to come from the output gap); and second, the coefficient estimate $\widehat{\kappa}$ is typically far from its theoretical level.

Blanchard and Gali (2005; BG, henceforth) present another criticism of the standard NKPC, which they term the "divine coincidence:" in most New Keynesian models, the two goals of stabilizing inflation and stabilizing the output gap do not conflict. This is strongly at odds with the predominant viewpoint amongst central bankers that there is a tradeoff of some sort; in other words, most believe that stabilizing inflation will, at least in some instances, lead to worsensing of the output gap. (Put differently, central bankers typically expect an output cost to disinflation.) These authors note that this coincidence is linked to a specific property of the standard model: the gap between the natural level of output and the *efficient* (first-best) level of output is *constant*, and invariant to shocks. Thus, if one stabilizes the output gap, one also stabilizes the *welfare-relevant* output gap. In turn, the divine coincidence property is traceable to the absence of non-trivial *real* imperfections in the standard model.

These considerations motivate BG to introduce a simple real imperfection: real wage rigidities.[4] In particular, they assume a partial adjustment model for wages, described below. They argue that such rigidities are a natural source of inflation inertia – indeed, they overcome the above-mentioned empirical weakness – and can gracefully account for the good empirical fit of traditional Phillips curve equations. Furthermore, this simple alteration of the standard New Keynesian model breaks

[4] Blanchard and Gali note that they are not the first to confront the divine coincidence and offer a resolution; there are at least two alternative approaches, one including distortion shocks of one form or another (e.g., Clarida, Gali and Gertler 1999, Clarida, Gali and Gertler 2001, Smets and Wouters 2003, Steinsson 2003), and the other exploring alternative wage and price setting behavior (e.g., Erceg, Henderson and Levin 2000).

the divine coincidence, bringing the model's performance more in line with the intuition of central bankers.

2.1 The BG Model

The BG model is standard along most dimensions, with two exceptions: the existence of the aforementioned real wage rigidity, and the existence of a non-produced input M to production. In particular, the BG model is summarized as follows.

Firms. There is the standard continuum of monopolistically competitive firms, each producing a differentiated product and facing isoelastic demand. The production function is given by

$$Y = M^{\alpha} N^{1-\alpha}$$

where Y is output, M is the non-produced input (with real price υ), and N is labor input. Real marginal cost is then given by

$$mc = w - mpn = w - (y - n) - \log(1-\alpha)$$

where w is the log of the real wage, taken as given by each firm.

Households. There is the standard large number of identical households, with time separable preferences, a constant discount factor β, and period-utility given by

$$U(C, N) = \log(C) - \exp\{\xi\} \frac{N^{1+\phi}}{1+\phi}$$

where C is the standard composite commodity (with elasticity of substitution between goods equalling ϵ), N is employment, and ξ is a potentially time-varying preference shock. The implied marginal rate of substitution (in logs) is given by

$$mrs = c + \phi n + \xi$$

Pricing. Price decisions are staggered in the standard Calvo manner, with θ being the fraction of firms not adjusting their price in any given period.[5] The evolution of real wages is assumed to

[5] This leads to the standard NKPC equation, valid in the neighborhood of a zero-inflation steady state: $\pi_t = \beta E \pi_{t+1} + \lambda(mc + \mu^p)$, where $(mc + \mu^p)$ is the log-deviation of real marginal cost from its value in a zero-inflation steady state, and $\lambda := \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

take the following form:

$$w_t = \gamma w_{t-1} + (1-\gamma)\, mrs$$

where γ can be interpreted as an index of real rigidities (driving a wedge between w and mrs). This "reduced-form" real rigidity is intended to parsimoniously model the slow adjustment of wages to labor market conditions. (In an appendix, BG formally model staggered real wage decisions, which makes the algebra more complex but does not alter their conclusions.) BG cite several other studies which have used a similar assumption, and note that the results in Fuhrer and Moore (1995)'s results stem from an assumption which leads to similar wage dynamics.

Inflation dynamics. Substituting and rearranging terms, BG obtain the following equilibrium inflation relation:

$$\pi_t = \beta E_t \pi_{t+1} + \frac{\lambda}{1-\gamma L} x_{2,t} \tag{2}$$

where x_2 is a linear combination of current and lagged output from its second-best, natural (flex-price) level y^*:

$$x_{2,t} = \frac{1}{1-\alpha}\left[(1-\gamma)(1+\phi)(y_t - y_t^*) + \gamma\alpha(\Delta y_t - \Delta y_t^*)\right]$$

Whither the divine coincidence? As in the standard NKPC equation, there is still an exact relation – albeit with slightly more complex dynamics – between inflation, expected inflation, and the output gap (now, both level and change). So fully stable inflation implies a fully stable output gap $(y - y^*)$. But now, *a fully stable output gap is no longer desirable.* Why? Because what matters for welfare is the gap $\left(y - y^1\right)$, where y^1 denotes the *first*-best level of output. But now, the gap between y^1 and y^* is not a constant, so stabilizing the output gap $(y - y^*)$ does not imply stabilizing the gap $\left(y - y^1\right)$.

Equation (2) also implies inflation inertia: any change in the output gap, even a purely transitory change, will have long-lived effects on inflation. The reason? Any change in the worker's reservation wage resulting from a change in output (and hence a change in employment) will affect the real wage – and hence, real marginal cost – only gradually. Therefore, the cost effect will outlive the change in output. The larger is γ, the greater the inertia.

2.2 Empirical analysis of the BG model

Following the development in BG, the previous point can be illustrated using a representation of inflation dynamics which is more closely linked to empirical inflation models found in the literature. Multiply both sides of (2) by $(1-\gamma L)$ to obtain

$$\pi_t = \frac{\gamma}{1+\beta\gamma}\pi_{t-1} + \frac{\beta}{1+\beta\gamma}E_t\pi_{t+1} + \frac{\gamma}{1+\beta\gamma}x_{2,t} + \zeta_t, \tag{3}$$

where $\zeta_t \equiv \left[\frac{\beta\gamma}{1+\beta\gamma}\left(\pi_t - E_{t-1}\pi_t\right)\right]$ is white noise.

Expression(3) is very similar to "hybrid" NKPC specifications, which include a lagged inflation term and which have been used in many empirical and policy analysis applications. In particular, it includes both backward-looking and forward-looking inflation terms, with coefficients whose sum is close to one. Thus, adding real wage rigidities can account for the observed persistence in inflation.

BG argue that it is inappropriate to attempt to directly estimate (3), since y^* is unobservable (a point stressed by Gali and Gertler 1999). Below, we confirm BG's suspicion of the ad-hoc measures of output gaps which have been used in the literature, measures which approximate the natural rate by smoothed functions of y. Indeed, we demonstrate below that these are theoretically inappropriate measures.

However, BG's model does imply an alternative representation of the inflation equation which can be directly taken to the data. In particular, define the *desired* quantity of labor supply (given current wage and marginal utility of income), n_t^*, implicitly as

$$w_t = y_t + \phi n_t^* + \xi_t$$

where the equilibrium condition $c = y$ has been imposed, and y (in general) differs from what it would be in the first-best equilibrium. (In the absence of real rigidities, there would be no involuntary unemployment, as wage is always equal to the marginal rate of substitution of households.) Accordingly, define the (involuntary) rate of unemployment, un_t, as the (log) deviation between the desired supply of labor and actual employment:

$$un_t := n_t^* - n_t$$

Given the real rigidities, BG derive[6] the following relation between real wage growth and the unemployment rate:

$$\Delta w_t = -\frac{(1-\gamma)\,\phi}{\gamma} un_t$$

A rate of unemployment above zero is associated with downward adjustment of real wages, and vice versa. The size of the adjustment is inversely related to γ (the size of the real rigidity) and positively related to ϕ (the slope of the labor supply). Then, after some lengthy manipulations (given in BG's appendix), one obtains the estimating equation

$$\pi_t = \frac{1}{1+\beta}\pi_{t-1} + \frac{\beta}{1+\beta}E_t\pi_{t+1} - \frac{\lambda\,(1-\alpha)\,(1-\gamma)\,\phi}{\gamma\,(1+\beta)}un_t + \frac{\alpha\lambda}{1+\beta}\Delta\upsilon_t + \zeta_t \qquad (4)$$

where $\Delta\upsilon_t$ is the change in the real price of the non-produced input. The error term ζ_t is still given by $\psi\,(\pi_t - E_{t-1}\pi_t)$, where ψ is a constant; hence white noise, orthogonal to all variables at $t-1$. BG note the similarity of (4) to the traditional Phillips curve specification.

Equation (4) can be estimated using instrumental variables. BG do so using annual data from 1960-2004. In their study, π_t is the percent change in the GDP deflator, un_t is the civilian unemployment rate, and $\Delta\upsilon_t$ is the percent change in the PPI raw materials index (relative to the GDP deflator). Their instruments are four lags of each of these variables. When they did not impose the theoretical restriction that the sum of the coefficients on lagged and expected inflation equals one, this specification yielded the following coefficient estimates (standard errors in brackets, constant not reported):

$$\pi_t = \underset{(0.09)}{0.66}\pi_{t-1} + \underset{(0.06)}{0.42}\pi^e_{t+1} - \underset{(0.08)}{0.20}un_t + \underset{(0.001)}{0.018}\Delta\upsilon_t + \zeta_t$$

These results are supportive of their model.

In contrast, estimation results using more recent, quarterly data fail to support this model. In particular, using the same procedure and the same variables (but at a quarterly frequency) over the 1984-2003 period (and including seasonal dummy variables, which are required in this quarterly regression), one obtains the following coefficient estimates (standard errors in brackets, constant and seasonal dummy variables not reported):

$$\pi_t = \underset{(0.15)}{0.32}\pi_{t-1} + \underset{(0.23)}{0.68}\pi^e_{t+1} + \underset{(0.06)}{0.02}un_t + \underset{(0.008)}{0.003}\Delta\upsilon_t + \zeta_t$$

[6] See derivation in Appendix C.

The estimated model suggests that there is no NKPC – i.e., changes in the unemployment rate are not related to changes in inflation. (Furthermore, as in BG, the estimated coefficient on π^e_{t+1} exceeds that on π_{t-1} – although this difference is not statistically significant.)

What accounts for this empirical failure? We believe it stems from a counterfactual implication of the BG (and other New Keynesian) models, namely that the relationship between inflation and the unemployment rate is a constant *across frequencies*, when in reality it is not. Thus, the (apparently insignificant) parameter estimate on un_t is actually an estimate of an average of distinct coefficient values at different frequencies. Below, we test the assumption of stability across frequencies, and find that the data reject it. Furthermore, once we allow the data to speak to the nature of frequency dependence in this relationship, we find evidence supporting the BG hybrid NKPC model. The pattern of frequency dependence detected is consistent with a natural rate interpretation, namely that lower frequency movements of the unemployment rate correspond to changes in the natural rate, and only these low-frequency deviations of the unemployment rate from its current natural rate have an impact on inflation. In other words, the data appear to suggest the following model of inflation:

$$\pi_t = \frac{1}{1+\beta}\pi_{t-1} + \frac{\beta}{1+\beta}E_t\pi_{t+1} - \frac{\lambda\,(1-\alpha)\,(1-\gamma)\,\phi}{\gamma\,(1+\beta)}\,(un_t - un_t^*) + \frac{\alpha\lambda}{1+\beta}\Delta v_t + \zeta_t$$

where the time variation in un_t^*, the natural rate of unemployment, roughly coincides with lower-frequency fluctuations in the unemployment rate. Thus, it appears that a further refinement of the BG model – i.e., one which features a time-varying natural rate – is necessary in order to account for the empirical relationship between inflation and the unemployment rate. A natural candidate is labor market search, as in, e.g., Nason and Slotsve (2004).

We leave the development of modifications to the BG model which could deliver such a time-varying natural rate to future work. Instead, we now review new econometric tools for quantifying frequency dependence in feedback relationships. Along the way, we review a demonstration of a key result which goes beyond the analysis of Phillips curves: standard methods for detecting and modeling frequency dependence fail when feedback is present in the relationship, as is the case in the inflation-unemployment relationship. This failure is due to the two-sided nature of the filtering used in these approaches to isolate a specific range of frequencies for analysis. Fundamentally, the two-sided filtering interacts with the feedback in the relationship to induce correlations between the filtered series and the relevant regression error terms, thus producing inconsistent parameter estimates. The methodology below is robust to this criticism.

3 Methodology

3.1 Characterizing frequency dependence

In Sections 3.1–3.3 we explain what frequency dependence is, what it is not, and why it makes a difference. Sections 3.4 and 3.5 discuss the Tan-Ashley approach to the detection and modeling of frequency dependence in the absence of feedback and its straightforward implementation in the time domain. Section 3.6 discusses the problematic nature of two-sided filtering in the context of feedback relationships and describes how we modify the Tan-Ashley methodology appropriately to deal with this problem. Section 3.7 addresses the issue of frequency band specification: how to select the number of frequency bands to consider, and the particular set of frequencies to be included in each band.

It is best to be clear at the outset as to the meaning of the term "frequency dependence" in the context of a regression coefficient. Consider the following aggregate consumption function:

$$c_t = \gamma_0 + \gamma_1 y_{t-1} + \gamma_2 y_{t-2} + \gamma_3 c_{t-1} + \varepsilon_t \tag{5}$$

where c_t and y_t are the deviations from trend of the log of aggregate consumption spending and disposable income in period t, and ε_t is a covariance-stationary error term. In this model γ_1 is the "short-run marginal propensity to consume," characterizing how consumption spending (on average) responds to fluctuations in y_{t-1}. In contrast, $\frac{(\gamma_1+\gamma_2)}{(1-\gamma_3)}$ is the "long-run marginal propensity to consume," the change in steady-state consumption from a one unit change in steady-state income; it answers the question, "How does average steady-state consumption spending vary across different steady-state after-tax income levels?" The distinction between γ_1 and $\frac{(\gamma_1+\gamma_2)}{(1-\gamma_3)}$ is *not* what we mean by frequency-dependence.

What we *do* mean by frequency-dependence is that, according to the permanent-income hypothesis, the value of γ_1 itself depends upon frequency. In particular, this hypothesis asserts that consumption should *not* change appreciably if the previous period's fluctuation in income is highly transitory (high-frequency), whereas consumption *should* change significantly if the previous period's fluctuation in income is part of a persistent (low-frequency) movement in income. γ_1, then, should be approximately equal to zero for high frequencies, and close to one for very low frequencies. Equation (5), in contrast, incorrectly restricts γ_1 to be the same across all frequencies.

This frequency dependence in γ_1 implied by the permanent income hypothesis concomitantly implies that γ_1 varies over time. For example, with adaptive expectations, the implication is that γ_1 will be larger if the deviation y_{t-1} has the same sign as the deviation y_{t-2}, so that the deviation y_{t-1} is part of a smooth pattern. Note that this dependence of γ_1 on the recent history of y_{t-1} (and the resulting frequency dependence in γ_1) can thus be viewed as a symptom of unmodeled nonlinearity in the relationship between c_t and y_{t-1}. This aspect of frequency dependence is discussed at some length in Tan and Ashley (1999a); see also Ashley and Verbrugge (2006b). Here, the essential point is that this frequency dependence in γ_1 further implies that the value of γ_1 is not a fixed constant; rather, it varies over time, due to its dependence on $y_{t-1}, y_{t-2}, y_{t-3}$, etc.

Similarly, viewing equation (5) as part of a bivariate VAR model, the impulse response function for c_t will be a function of past innovations in both equations, and c_t will depend differently on different lags in the y_t innovations. Frequency dependence alters the nature of the impulse response functions. In particular, if there is no frequency dependence in the $c_t - y_t$ relationship, then the moving average representation of the c_t process will be a linear function of serially independent innovations; this leads to a set of conventional *linear* impulse response functions in which the change in the expected value of c_{t+n} induced by an innovation in the y_t process of size δ is unrelated to the values of previous innovations. Conversely, frequency dependence in the $c_t - y_t$ relationship implies that the full moving average representation of the $c_t - y_t$ relationship (and hence, the impulse response functions also) are *nonlinear* functions of serially independent innovations. Thus, in that case, the change in the expected value of c_{t+n} induced by an innovation in the y_t process of size δ *does* depend on the values of previous innovations. (Of course, the Wold Theorem still guarantees the existence of a linear MA(∞) representation for c_t and y_t – and hence of a set of linear impulse response functions for these variables – but the innovations in *this* linear MA(∞) representation are *not serially independent*.)

The following explicit example clarifies this point.[7] Consider the particular case in which the linear moving average (Wold) representation for a series c_t can be approximated by the MA(1) process:

$$c_t = v_t + \gamma_1 v_{t-1}$$

in which the v_t innovation series is generated by the bilinear process:

$$v_t = 0.7 v_{t-2} u_{t-1} + u_t$$

where u_t is serially independent. It is easy to verify that the v_t generated by this bilinear process are serially uncorrelated, so this MA(1) process could in principle be the Wold representation for c_t. Now rewrite the moving average representation of c_t as a function of the current and past values of the serially independent innovations – $u_t, u_{t-1}, \ldots$ – by repeatedly substituting the bilinear model in to eliminate v_t, v_{t-1}, etc. from the model for c_t. In this way one obtains:

$$c_t = u_t + (\gamma_1 + 0.7u_{t-2} + \text{higher order terms})\, u_{t-1} + (0.7\gamma_1 u_{t-3} + \text{higher order terms})\, u_{t-2} + \ldots$$

where the higher order terms involve $(0.7)^2\, v_{t-4}u_{t-3}$, $(0.7)^2\, v_{t-5}u_{t-4}$, and so forth. Continued substitution would further elaborate these terms, but the point is clear: the coefficient on the serially independent innovation u_{t-1} is no longer a constant. Instead, it is $(\gamma_1 + 0.7u_{t-2})$ plus higher order terms. Consequently, the impulse response function at lag one is frequency dependent in the sense discussed here: the coefficient on u_{t-1} will be different when the previous innovation (u_{t-2}) is of the same sign as u_{t-1}. Thus, estimating a linear moving average model for c_t yields an impulse response coefficient estimate at lag one which cannot be stable over time or across frequencies, since c_t responds differently to a lag-one shock which is part of a smooth pattern than to a lag-one shock which has just changed sign from the previous period.

Finally, we conclude this section with a warning from McCallum (1984): "...the association of low-frequency time series statistics with 'long-run' economic propositions is not generally warranted. Instead, many so-called long-run relationships involve expectational relationships which have little or nothing to do with frequencies per se." Thus, any conclusion about the low frequency behavior of a model parameter, such as γ_1 in equation (5) is best viewed as an assertion as to how c_t responds

[7] See Potter (2000) for a formal treatment of nonlinear impulse response functions.

to smooth fluctuations in y_{t-1}, not as a statement with regard to the long run relationship between c_t and y_t.

3.2 Consequences of frequency dependence

Now consider a simple bivariate time series model:

$$y_t = \beta x_t + \varepsilon_t \qquad \varepsilon_t \sim NIID\left[0, \sigma^2\right]$$

for $t \in \{1, ...T\}$. The parameter β can be interpreted as $dE[y_t|x_t]/dx_t$; if β actually takes on two values – β_0 in the first half of the sample and β_1 in the second half of the sample, for example – then this regression is clearly mis-specified. In that case, the usual statistical machinery for testing hypotheses about β is invalid – indeed, the hypotheses themselves are essentially meaningless, since β does not have a single well-defined value to test. Similarly, the least-squares estimate of β is clearly neither a consistent estimator for β_0, nor for β_1. In particular, if the sign of the relationship is positive in the first part of the sample and negative later on, then the least squares estimate of β might well be close to zero, leading to the erroneous conclusion that y_t and x_t are unrelated.

One of the key implications of the spectral regression model of Engle (1974, 1978) – summarized in section 3.3 below – is that β is stable across time if and only if it is stable across *frequencies*; this was also discussed in the context of the simple consumption function example in the previous section. Thus, if the value of β is different at low frequencies than at high frequencies, then β varies over time also, albeit in a manner which might be difficult to detect with time domain parameter stability tests. Still, this result implies that frequency variation in β yields all of the same unhappy properties as does time variation. In particular, the least squares estimator of β is an inconsistent estimator of $dE[y_t|x_t]$ with respect to x, and – since β does not have a unique value – hypothesis tests about β are of doubtful value.

Frequency dependence in the unemployment rate coefficient of equation (4) might arise from mis-specified dynamics for the natural rate; or it could occur for other reasons. We take such frequency dependence to be an empirical issue – one which is consequential for the foregoing reasons – and below develop methods for detecting and correcting for it.

3.3 Pseudo frequency dependence

It is important to distinguish 'true' frequency dependence in a relationship from a superficially similar concept in which the coefficients of the model quantifying the relationship are constant, but the *coherence* (closely related to the magnitude of the cross-spectrum of the variates) is frequency-dependent. This latter notion is used in Geweke (1982), Diebold, Ohanian and Berkowitz (1998), and a host of other studies. These decompositions are mathematically sound, but we call what they measure 'pseudo frequency dependence' because – since the underlying model coefficients are assumed constant – such measures do not actually quantify frequency variation in the relationship itself.

A simple example clarifies this distinction. Consider the following consumption relation,

$$c_t = \beta y_{t-1} + u_t + \phi u_{t-1}$$

$$\begin{pmatrix} u_t \\ y_t \end{pmatrix} \sim NIID \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \right]$$

The marginal propensity to consume in this relationship is clearly a constant (β) and Fourier transforming both sides of this equation will do nothing to change that – it merely yields a relationship between the Fourier transform of c_t and the Fourier transform of y_{t-1}, still with a constant coefficient β. (E.g., see Section 3.4 below.) But the cross-spectrum and coherence functions relating c_t and y_t are *not* constants: by construction, they depend explicitly upon the frequency parameter ω. In particular, Geweke (1982)'s measure of the strength of the linear dependence of c_t on y_{t-1} (a generalization of the coherence function) for this model is:

$$f_{y \to c}(\omega) = \frac{1}{2} \ln \left\{ \frac{\sigma_u^2 \left(1 + \phi^2 - 2\phi \cos(\omega)\right) + \beta^2 \sigma_y^2}{\left[\sigma_u^2 \left(1 + \phi^2 - 2\phi \cos(\omega)\right)\right]^2} \right\}$$

which clearly *does* depend upon frequency so long as the moving average parameter ϕ is not zero.

Evidently, this frequency dependence in Geweke's measure (and in the other 'strength of association' measures based upon the cross-spectrum and the coherence function) is not quantifying the frequency variation in the c-y relationship itself, since there is none to quantify. So what <u>*is*</u> it doing? These kinds of measures are usually interpreted as quantifying the degree to which the overall R^2 for the equation is due to sample variation at low frequencies versus high frequencies.

Suppose that ϕ is positive, in which case Geweke's measure indicates that low frequencies are important to the R^2 of the relationship. This says nothing about whether consumption and income are differently related at low versus high frequencies – that depends upon the marginal propensity to consume (β), which is constant. Rather, it says that this dynamic relationship transforms serially uncorrelated fluctuations in y_{t-1} and u_t into positively correlated fluctuations in c_t. Alternatively, one could observe that c_t in that case has substantial spectral power at low frequencies, and interpret this result, to paraphrase Geweke (1982, p. 312), as indicating that the white noise innovations in y_{t-1} explain most of this low frequency portion of the variance in c_t.[8]

3.4 Regression in the frequency domain in the absence of feedback

The most elegant way to assess the actual frequency dependence of a regression coefficient is to estimate the regression equation in the frequency domain. Such spectral regression was originally proposed by Hannan (1963) and most clearly exposited in Engle (1974, 1978). Following Engle, spectral regression is based on the simple notion that a multiple regression model in the time domain, such as

$$Y = X\beta + \varepsilon \qquad \varepsilon \sim N\left[0, \sigma^2 I\right] \tag{6}$$

can be Fourier-transformed on both sides of the equation via multiplication by a complex-valued matrix W, yielding

$$\begin{aligned} WY &= WX\beta + W\varepsilon & (7) \\ \widetilde{Y} &= \widetilde{X}\beta + \widetilde{\varepsilon} \qquad \widetilde{\varepsilon} \sim N\left[0, \sigma^2 I\right] & (8) \end{aligned}$$

where $\widetilde{Y} = WY$, etc., and where the $(j,k)^{th}$ element of W is given by $w_{jk} = \frac{1}{\sqrt{T}} \exp\left(\frac{2\pi ijk}{T}\right)$, with T equal to the sample length. The variance of $\widetilde{\varepsilon}$ is still $\sigma^2 I$ because W is an orthogonal matrix.

Note that the coefficient vector β is identical in both equation (6) and equation (8). What *has* changed, however, is that the T sample observations in Y and in each column of X are replaced by T observations on each $\widetilde{Y}$ and each column of $\widetilde{X}$, each of which now corresponds to a frequency in the

[8] Note also that both the coherence and gain functions are, by construction, non-negative at all frequencies. Thus, neither of these concepts can possibly capture frequency dependence as discussed here, which can readily involve a regression coefficient having one sign at low frequencies and the opposite sign at high frequencies.

interval $[0, 2\pi(T-1)/T]$. In particular, one can identify the j^{th} 'observation' in this transformed regression model as corresponding to frequency $2\pi(j-1)/T$.

Note, however, that consistent least squares estimation of β in equation (8) requires that $corr(\widetilde{x}_{k,j}, \widetilde{\varepsilon}_j)$ is zero for all values of j and k, where $\widetilde{x}_{k,j}$ is used to denote the j^{th} observation on $\widetilde{x}_k$. Since W embodies a two-sided transformation – i.e., $\widetilde{x}_{k,j}$ depends upon all of $x_{k,1}, ..., x_{k,T}$ and $\widetilde{\varepsilon}_j$ depends upon all of $\varepsilon_1, ..., \varepsilon_T$ – this condition requires that $x_{t,k}$ be uncorrelated with both past and future values of ε_t. This issue is taken up more explicitly in Section 3.6 below; it is side-stepped here by restricting attention to relationships in which there is no feedback between y_t and $x_{k,1}, ..., x_{k,T}$.

Spectral regression has unique advantages over regression in the time domain. For example, missing observations and distributed lag expressions involving non-integer lags can be dealt with fairly readily in the frequency domain. And – vital for the present context – detecting and modeling frequency variation in a component of β corresponds precisely to testing for instability in this component across the sample observations in equation (8).

Prior to Tan and Ashley (1999), however, this framework also had some fairly intense drawbacks, which severely limited its usefulness and acceptance. For one thing, $\widetilde{Y}$ and $\widetilde{X}$ are complex-valued, precluding the use of ordinary regression software to estimate β. An estimator for β can be expressed in terms of the cross-periodograms of Y and the columns of X – e.g., equation 10 of Engle (1974) – but the calculations still require specialized software. Consequently, Engle's approach is really only convenient for considering parameter variation over at most two frequency bands: in that special case it is possible to finesse the problem so that ordinary regression software suffices.[9]

Another problem with Engle's framework is really just cosmetic, but nevertheless effectively limits the credibility of the results: one cannot drop a group of, say, the five lowest-frequency observations without also dropping the five observations at the highest five frequencies – otherwise, the least squares estimate of β is no longer real-valued. These latter five observations, at what appear to be the five highest frequencies, in fact actually do correspond to low frequencies because of symmetries in the W matrix, but one is apt to lose one's audience in trying to explain it.

[9] Later work by Thoma (1992, 1994) pushes this idea a bit further by observing how the parameter estimate varies as more frequencies are added to the low frequency band.

Finally, Engle's formulation does not deal with econometric complications such as simultaneity, cointegration, or feedback. Phillips (1991) provides a framework for estimating cointegrated systems in the frequency domain based directly on Hannan's formulation in terms of the spectra and cross-spectra of the data. But this approach again requires specialized software, and is still applicable only to non-feedback relationships.

The net result is that spectral regression methods have been applied to the frequency dependence problem for only a handful of macroeconomic relationships.

The approach developed in Tan and Ashley (1999) effectively eliminates the objections noted above, at least for non-feedback relationships. This formulation is similar in spirit to Engle's except that the complex-valued transformation matrix (W) is replaced by an equivalent *real*-valued transformation matrix (A) with $(j,t)^{th}$ element:

$$a_{j,t} = \begin{cases} \frac{1}{\sqrt{T}} & j = 1 \\ \sqrt{\frac{2}{T}} \cos\left[\frac{\pi j(t-1)}{T}\right] & j = 2, 4, ..., (T-2) \text{ or } (T-1) \\ \sqrt{\frac{2}{T}} \sin\left[\frac{\pi (j-1)(t-1)}{T}\right] & j = 2, 4, ..., (T-1) \text{ or } T \\ \frac{1}{\sqrt{T}} (-1)^{t+1} & j = T \text{ and } T \text{ is even, } t = 1, ..., T \end{cases} \tag{9}$$

This transformation, which first appears in Harvey (1978), yields a real-valued frequency domain regression equation

$$AY = AX\beta + A\varepsilon \qquad A\varepsilon \sim N\left[0, \sigma^2 I\right]$$

or

$$Y^* = X^*\beta + \varepsilon^* \qquad \varepsilon^* \sim N\left[0, \sigma^2 I\right] \tag{10}$$

with $Y^* = AY$, etc. In fact, each row of A is just a linear combination of two rows in the W matrix, based on the usual exponential expressions of the sine and cosine – e.g., $\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$. Again, $Var(\varepsilon^*) = Var(\varepsilon)$ because A is an orthogonal matrix.

Since the elements of the A matrix are all real-valued, equation (10) can be estimated using ordinary regression software. Moreover, the effect of the transformation on a column vector (e.g., Y) is now plain to see. The second and third rows of the A matrix ($j = 2$ and 3) correspond to the two observations at the lowest non-zero frequency. The weights in these rows make one complete oscillation over the T periods in the actual sample, so any fluctuation in Y_t that is sufficiently brief as to average out to essentially zero over a period of length $T/2$ will have little impact on either Y_2^* or Y_3^*. In contrast, suppose that T is even and consider the highest frequency row of A. This row simply averages $T/2$ changes in the data; clearly, it is ignoring any slowly-varying components of the data vector and extracting the most quickly-varying component.

The observations in this regression model thus do correspond to frequencies. Consequently, frequency variation in, say, β_k – the k^{th} component of β – can be assessed by applying any of the variety of procedures in the literature for examining the variation in an estimated regression coefficient across the sample observations: e.g., Chow (1960), Brown, Durbin and Evans (1975), Farley, Hinich and McGuire (1975), Ashley (1984), or Bai (1997) and Bai and Perron (1998, 2003). We will return to this issue in Section 3.5; for now, we observe that Tan and Ashley (1999) use the procedure given in Ashley (1984) and simply partition the T observations in equation (10) into m equal frequency bands and estimate how β_k varies by replacing X_k^*, the k^{th} column of X^*, with m appropriately constructed dummy variables:[10]

$$Y^* = X_{\{k\}}^* \beta_{\{k\}} + D^* \gamma^* + \upsilon^* \tag{11}$$

where $X_{\{k\}}^*$ is X^* omitting the k^{th} column, and $\beta_{\{k\}}$ is β omitting the k^{th} component. The columns $\left[D^{*1}...D^{*m}\right]$ comprising the D^* matrix consist of m new explanatory variables, one for each frequency band: D_j^{*s}, the j^{th} component of the new explanatory variable for frequency band s, is zero for each component outside the frequency band, and equal to the corresponding component of X_k^* (the k^{th} column of X^*) for each component inside the frequency band.[11]

3.5 The Tan-Ashley approach in the time domain

It is both helpful and instructive to re-cast the Tan-Ashley formulation in the time domain. Since A is an orthogonal matrix, A^{-1} is just its transpose, A^T. Multiplying the regression model of (11)

[10] Simulations in Ashley (1984) indicate that this modest generalization of the Chow test performs at least as well as more sophisticated alternatives with samples of moderate length.

[11] We note that the foregoing analysis could all be applied substituting *any* orthogonal matrix for A, so long as its rows pick out components of increasing smoothness. The finite Fourier transforms used here are both familiar and compellingly unique, so long as one assigns the same coefficient value to both the sine and cosine rows corresponding to a particular frequency. But this is not to say that a useful transformation matrix could not be formulated in other ways – e.g. using wavelets, as defined defined by Ramsey and Lampert (1998a,b).

through by A^T yields

$$A^T Y^* = A^T X^*_{\{k\}} \beta_{\{k\}} + A^T D^* \gamma + A^T \upsilon^* \tag{12}$$

and hence

$$Y = X_{\{k\}} \beta_{\{k\}} + D\gamma + \upsilon \tag{13}$$

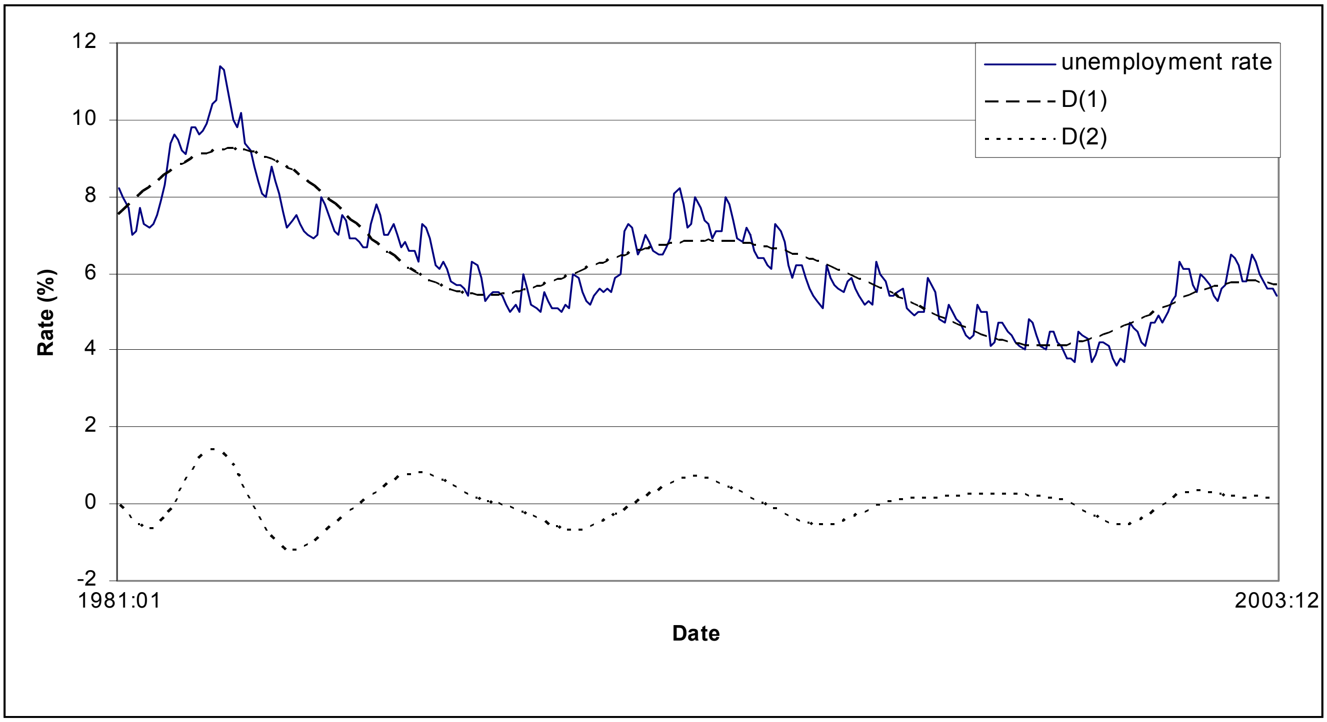
Here Y is the original dependent variable data vector and $X_{\{k\}}$ is the original data matrix, omitting the k^{th} column.

The matrix $D = \left[D^1 ... D^m\right]$ thus has as its columns the back-transforms of the frequency-domain explanatory variables $\left[D^{*1} ... D^{*m}\right]$ corresponding to each of the m frequency bands being considered. Note that, since the columns $\left[D^{*1} ... D^{*m}\right]$ are orthogonal and add up to $X^*_k = AX_k$, the column vectors comprising $\left[D^1 ... D^m\right]$ are orthogonal also and add up to X_k, the original data vector for the k^{th} explanatory variable.[12] Consequently, the error vector υ is identical to the original error term in (6) if the m components of γ are all equal to β_k.

The column vectors $\left[D^1 ... D^m\right]$ are in essence bandpass filtered versions of X_k which partition this variable into m orthogonal components, one for each frequency band. For example, suppose that one were to partition the monthly US unemployment rate into three frequency components: D^1_t, comprising the fluctuations corresponding to low frequencies (periods greater than 60 months); D^2_t, a medium-frequency ("business cycle") component, corresponding to periods between 18 and 60 months; and D^3_t, a high-frequency component, corresponding to periods less than 18 months. Figure 1 plots the monthly US unemployment rate, along with D^1_t and D^2_t – the first and second of these components – using data from 1980 through 2003.

[12] Tan and Ashley (1999) give an explicit example of this with $m = 3$ frequency bands. Given their particular partitioning , they show how D^{*1} is zero except for the first third of the observations (corresponding to the lowest frequencies) – yielding a smooth D^1 time domain series – whereas D^{*3} is zero except for the last third of the observations (corresponding to the highest frequencies), and yields a rapidly varying D^3 time domain series. They do not, however, point out that the m filtered components $\left[D^1 ... D^m\right]$ are orthogonal.

Figure 1: Time Plot of the US Unemployment Rate and its Low- and Medium-Frequency Components (D_t^1 and D_t^2)



No one of these m implied bandpass filters is an optimal bandpass filter. One might choose a Baxter-King (1999) or Christiano-Fitzgerald (2003b) bandpass filter for that purpose. But $\left[D^1...D^m\right]$ have the desirable property of partitioning X_k in an intuitively appealing way into m orthogonal components that add up exactly to X_k. Consequently, replacing $\beta_k X_k$ by $D\gamma$ in the regression equation allows one to conveniently test for, and model, frequency dependence in β_k, with frequency stability corresponding to the null hypothesis that all m components of γ are equal.

In contrast, note that failing to replace $X_k \beta_k$ by $D\gamma$ when the m components of γ are *not* equal yields a mis-specified regression model for Y: $\widehat{\beta}_k^{OLS}$ cannot possibly be consistent for β_k in this model since β_k does not in that case have a unique value to estimate.

Note also that, since X_k equals $D^1+...+D^m$, replacing $X_k \beta_k$ by $D\gamma$ in a regression model leaves the properties of the error term unaffected under the null hypothesis of no frequency dependence.

No sample information is lost; the only statistical cost is a loss of $m-1$ degrees of freedom, since more coefficients are being estimated. In contrast, a typical bandpass filtering analysis – e.g., Christiano and Fitzgerald (2003a), Comin and Gertler (2003), Den Haan and Sumner (2004), or Malin (2006) – applies a bandpass filter to both Y and to some of the columns of X. Thus, one ends up in such analyses quantifying the relationship between these filtered time series, rather than the relationship between the actually observed variables. Such analyses also require an *a priori* selection of the frequency band to consider, which (as is described below) our approach does not.

Finally, note that there is nothing essential about the simple form of the original model ($Y = X\beta+\varepsilon$) in the analysis above. One could just as easily use this approach to investigate the frequency dependence of the coefficient on X_k in more complex settings by replacing $X_k\beta_k$ with the weighted sum $D\gamma$ *regardless of how* X_k *enters the analysis* - linearly or nonlinearly, instrumented or not, etc. – using essentially the same econometric techniques and software one was already employing.

3.6 The Problem with Feedback – and a Solution using One-Sided Filtering

Note that $\widehat{\gamma}^{OLS}$ will be a consistent estimate of γ in equation (13) if and only if the error term in this equation is uncorrelated with each of the regressors $D^1...D^m$. Since the t^{th} observation on each of these regressors is the result of what amounts to a two-sided nonlinear bandpass filter applied to $X_{k,t}$, this will be the case only if X_k is strongly exogenous, that is, only if every observation on X_k is uncorrelated with every observation on the error term in the original regression model. (This is, of course, equally the case for *any* methodology which applies a two-sided bandpass filter to the k^{th} regressor.) Unfortunately, feedback in the relation between Y_t and $X_{k,t}$ induces exactly this kind of correlation.

For example, consider the analysis of possible frequency dependence in the parameter λ_2 of the following bivariate equation system:

$$\begin{aligned} y_t &= \lambda_1 y_{t-1} + \lambda_2 x_{t-1} + \varepsilon_t \\ x_t &= \alpha_1 x_{t-1} + \alpha_2 y_{t-1} + \eta_t \end{aligned} \tag{14}$$

Clearly, this is a feedback relationship only if α_2 is nonzero. But note that the x_t equation implies that

$$
\begin{aligned}
x_t &= \alpha_1 x_{t-1} + \alpha_2 y_{t-1} + \eta_t \\
&= \alpha_1 x_{t-1} + \alpha_2 \left(\lambda_1 y_{t-2} + \lambda_2 x_{t-2} + \varepsilon_{t-1}\right) + \eta_t \\
&= \alpha_1 x_{t-1} + \alpha_2 \lambda_1 y_{t-2} + \alpha_2 \lambda_2 x_{t-2} + \alpha_2 \varepsilon_{t-1} + \eta_t
\end{aligned}
$$

so that x_t is correlated with ε_{t-1} if there is feedback from past y_t to x_t. But, two-sided filtering implies that x^*_{t-1} depends upon x_t, x_{t+1}, x_{t+2}, etc., so that x^*_{t-1} is thus correlated with $\varepsilon_{t-1}, \varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+1}, ...$, which (under two-sided filtering) are correlated with ε^*_t. Thus, in the presence of feedback, a two-sided transformation of x_{t-1} will in general produce a transformed explanatory variable, x^*_{t-1}, which is correlated with the transformed error term, ε^*_t, yielding inconsistent least-squares parameter estimates. (Examples of such two-sided filters include a filter based on the A matrix as discussed above, or the Hodrick-Prescott (1987) filter, or bandpass filters such as those given by Baxter and King (1999) or Christiano and Fitzgerald (2003).)

To eliminate this problem, we exploit the fact that the Tan-Ashley formulation is easily adapted to use only one-sided filtering.[13] The modified calculation steps through the sample using blocks of length τ. In the first step, observations one through τ on X_k (i.e., $X_{k,1}, ..., X_{k,\tau}$) are used to compute the τ-dimensional column vectors $D^1...D^m$, one for each of the m frequency bands. The last (period τ) element in each of these vectors becomes the period τ observation on $D^1...D^m$ for use in estimating equation (13). Next one uses the τ sample observations $X_{k,2}, ..., X_{k,\tau+1}$ to re-compute the τ-dimensional column vectors $D^1...D^m$. Again the last (τ^{th}) element in each of these column vectors becomes the period τ+1 observation on $D^1...D^m$ for use in estimating equation (13). And so forth.[14] Thus, one could characterize $D^1...D^m$ as being the result of a set of m one-sided bandpass filters obtained using a moving block of τ observations.

The resulting $D^1...D^m$ columns still add up precisely to the original explanatory variable (X_k) over its last $T - \tau$ elements. These m columns are no longer orthogonal, but in practice they are not highly correlated with one another. In any case, the orthogonality is of modest importance:

[13] Christiano and Fitzgerald (2003b) also provide a one-sided version of their filter, but they do not propose stepping this filter through the sample data, as is discussed below. Also, as noted above, their filter does not have the desirable property of being able to partition X_k into m components that add up exactly to X_k.

[14] Windows-based, and RATS, software implementing this partitioning of a given input column vector is available from the authors.

what is essential is that $D^1 ... D^m$ still precisely partition (sum up to) X_k and that they are now the product of a one-sided filter.

One must lose the use of $\tau - 1$ start-up observations in estimating equation (13) in this way, but this is necessary in order to avoid spurious results when feedback is present. This loss is analogous to the start-up observations lost in using lagged variables in an equation. This loss in degrees of freedom is manageable in the Phillips curve application given in Section 5 below: we use five years of pre-sample data – 1979-1983 – so as to be able to consider frequencies corresponding to periods as large as sixty months.

Lastly, it must be mentioned that bandpass filters like the ones used here generically have problems near the endpoints of the sample. The standard method for addressing this shortcoming – as, for example, in Stock and Watson (1999) – is to augment the sample using projected values obtained from univariate autoregressive models.[15] Christiano suggests using projections from multivariate models (private communication, 2005). Here, building upon this suggestion, we adopt a method which has found wide acceptance in the forecasting literature: forecast combination (see, e.g., the review in Timmermann 2005). In particular, we forecast the unemployment rate using a simple average[16] of three forecasts: one from an AR(4) model of the unemployment rate (with seasonal dummy variables), one from an ARMA(5,1) model of the unemployment rate, and one from a multivariate model which included lags of the unemployment rate, seasonal dummy variables, detrended industrial production, unemployment insurance extensions, and the index of help-wanted advertising. Each model is estimated using only observations through the last observation in the window (observation τ), and used to forecast the series for an additional four quarters. The resulting (averaged) $\tau + 4$ observations are then decomposed into the m frequency components, and the τ^{th} observation on each component is used to produce the values of $D^1 ... D^m$ from this window. The $D^1 ... D^m$ column vectors produced in this way still (by construction) add up precisely to X_k; they are still each the product of an entirely one-sided bandpass filter; and (since their values are now no longer close to the endpoint of each window) they produce quite satisfactory decompositions.[17]

[15] The idea of improving filtering via augmentation of data using forecasts originated in Dagum (1978).

[16] Interestingly, a common finding in the literature on forecast combination is that equal-weighted forecasts perform quite well and are difficult to beat; see, e.g., Clemen (1989) and Stock and Watson (2001).

[17] It would seem advisable to detrend the X_k data in each window, since a somewhat persistent time series can appear quite trended in each of the sequence of windows, even though it is not trended overall. Thus, a linear trend is estimated over the $\tau + 4$ observations in each window, and subtracted from the X_k values prior

Ashley and Verbrugge (2006a) present simulation evidence which demonstrates the efficacy of this procedure. In particular, the simulation results reported address three questions relating to data-generating processes which feature feedback. First, in the presence of such feedback, does two-sided filtering actually lead to a spurious finding of frequency-dependence when none actually exists? Second, does the one-sided procedure proposed in Section 3.5 avoid such spurious findings? Finally, does the one-sided procedure correctly detect, and appropriately model, frequency-dependence when such dependence is present? Each question is answered in the affirmative. The conclusion is that the procedure described above is both necessary and effective in the presence of feedback.

3.7 Frequency Band Specification

Selecting the number of frequency bands, and the particular set of frequencies to be included in each band, is an important issue in implementing the analysis described above.

One approach is to specify m bands on *a priori* grounds; this is analogous to common practice in empirical macroeconomics, where attention is often restricted to "business cycle" frequencies. In the present context, this "calendar-based" approach might suggest a three-band formulation – one band containing all frequencies corresponding to periods of less than, say, 6 quarters, a second band containing frequencies corresponding to periods between $1\frac{1}{2}$ and 5 years, and a third containing all frequencies corresponding to longer periods. This choice seems reasonable, but it is quite *ad hoc*: one might equally well choose one of many other calendar-based frequency band structures. Furthermore, one risks faulty inference. If the chosen calendar-based bands are consistent with the actual pattern of frequency dependence present in the data, then this procedure will have high power to detect that pattern. But if not, then the calendar-based test could have relatively low power: One might unnecessarily fail to uncover an existing pattern of frequency dependence in a particular regression coefficient through a maladroit selection of a calendar-based frequency band structure. Moreover, even if one does still detect frequency dependence in spite of such a maladroit choice, the pattern of frequency dependence thus observed will surely be distorted to some degree.

An alternative approach is to choose the number and composition of the frequency bands so as

to decomposing it into the m frequency components. After the decomposition is performed, observation τ's estimated trend value is then added back into observation τ of the lowest frequency band, D^1. In this way, $D^1 ... D^m$ still sum to X_k.

to minimize an adjusted goodness-of-fit criterion, such as the Bayes-Schwarz Information Criterion (BSIC). However, in this case the sampling distribution of the F statistic for testing the null hypothesis of equal coefficients on all bands must be obtained by simulation so as to properly account for the extensive specification search undertaken; unfortunately, this leads to a test of very low power.

The approach adopted here is to simply allow the regression equation to estimate a distinct coefficient for every possible frequency allowed by the limited length of the window used to implement the one-sided filtering. For example, with the 24-quarter windows used in the Phillips Curve model estimated in Section 5 below, only 12 distinct frequencies – listed in Appendix 1 – are possible.[18] With, in this case, 80 sample observations, estimating 13 frequency band coefficients is not prohibitively costly in terms of degrees of freedom.[19]

4 Estimation Results

4.1 Regression model specification

As noted above, we take our specification from Blanchard and Gali (2005):

$$\pi_t = \alpha + \phi\pi_{t-1} + (1-\phi)\, E_t\pi_{t+1} + \beta un_t + \theta\Delta\upsilon_t + \xi_t \tag{15}$$

Following Blanchard and Gali, we use as our measure of inflation the (quarterly) GDP deflator, and use inflation in the PPI raw materials index (relative to the GDP deflator) as the measure of "supply shocks" $\Delta\upsilon_t$; we also include seasonal dummies. un_t is the non-seasonally-adjusted total civilian unemployment rate. We estimate Equation (15) over the period 1984:I-2003:IV using instrumental variables, with both un_t and π^e_{t+1} being treated as endogenous; instruments are lags of each regressor, as noted below.

Using the one-sided filtering methodology described in Section 3.5 above, the series un_t was

[18] There are half as many frequencies as months in the window because there is both a sine and a cosine row in the A matrix of Section 3.4 for each distinct frequency.

[19] It does seem a bit wasteful, however, in view of the fact that one expects the frequency variation across frequencies to be fairly smooth. Consequently, we also investigated a more parsimonious approach in which the variation of the 13 coefficients is modeled by means of a Chebyshev polynomial; we obtained similar results.

decomposed into frequency bands $un^1 ... un^k$, with k chosen as discussed below. Equation (15) was estimated in the form[20]

$$\pi_t = \alpha + \phi_1 \pi_{t-1} + \phi_2 \pi^e_{t+1} + \sum_{j=1}^{k} \beta_j un^j_t + \theta \Delta v_t + \xi_t \tag{16}$$

using instrumental variables; instruments were four lags of each regressor.[21]

We performed two alternative tests of frequency dependence based on Equation (16). In one case, un_t was partitioned into three components using "calendar-based" frequency bands, as detailed below. In the other, un_t was fully partitioned into 13 components[22] – one for each distinct frequency in a 24-quarter rolling window. Frequency-independence is rejected if the null hypothesis $H_0 : \beta_i = \beta_j, \forall i \neq j$.

As noted in Section 3.7, calendar-based bands are rather *ad hoc*. Consequently, unless one has a specific and strongly-held prior opinion as to what frequency band structure is consistent with any actual frequency dependence, then the "full-partition" procedure is more appropriate. (And, of course, basing one's calendar-based bands upon the results of a specification search and pretending that this search activity did not take place yields a test of unknown size.)

Here, for illustrative purposes, results are given using both approaches. Three calendar-based bands are specified: a high-frequency band (periods less than 6 quarters), a "business-cycle" band (periods between 6-12 quarters), and low-frequency band (periods greater than 12 quarters).[23]

Setting the window length τ to a number larger than 20 quarters – corresponding to using more than five years of data at the start of the sample in constructing the first window – seemed unreasonable, given the length of our sample period. Consequently, τ was set to five years, implying

[20] Additional lags of the unemployment rate were not significant. The distribution of ε_t appears to be somewhat thick-tailed, with a kurtosis test p-value of 0.03. Consequently, the externally-studentized residual test (which is equivalent to many outlier tests – see Verbrugge 2005) yields seven apparently-significant fitting errors. Inclusion of dummy variables for these outlying observations does not materially change the significance or pattern of the frequency dependence found below, however.

[21] In the full-partition case, this resulted in too many instruments; thus, in that case, we used only two lags of each component of un_t.

[22] The Appendix lists the frequencies and periods associated with a 24-observation rolling window. Recall from the discussion at the close of Section 3.6 that the 20 quarters of actual data ($un_{t-19} ... un_t$) are augmented by four quarters of projected data, so that the filtered value for each frequency band is four quarters prior to the end of a 24-quarter filtering window.

[23] Our filtering window length does not allow fluctuations with periods between 12 and 24 quarters in length to be distinguished (see Appendix A); this rules out the use of a more conventional "business-cycle" band with periods between 6 and 20 quarters. As a robustness check, we also used a six-year window, which allowed a "business-cycle" band of 5.6-14 quarters; inferences were similar.

that each un_t^j observation is based on five-years' worth of prior data, and (including the 4-quarter projection discussed at the end of Section 3.6) implying that the filtering window is 24 quarters long.

4.2 Results

Estimating the BG hybrid New Keynesian Phillips curve specification of Equation (15) – i.e., not allowing for possible frequency dependence in β – over the sample period 1984:1-2003:4 yields the IV estimates:[24]

$$\pi_t = \underset{\underset{(-0.49)}{-0.67}}{\alpha} + \underset{\underset{(0.15)}{0.32}}{\phi_1}\pi_{t-1} + \underset{\underset{(0.23)}{0.68}}{\phi_2}\pi_{t+1}^e + \underset{\underset{(0.06)}{+0.02}}{\beta}un_t + \underset{\underset{(0.008)}{-0.003}}{\theta_1}\Delta\upsilon_t + \underset{F-test:\ p=0.000}{\sum_{i=1}^{3}\theta_{i+1}quarter_t^i} + \xi_t \qquad (17)$$

Coefficient estimates, with their estimated standard errors in parentheses, are given above for most coefficients; but for the quarterly dummy variables $quarter_t^i$, we simply present the p-value for the F-test of the null hypothesis that all three coefficients are zero. Here, $\Delta\upsilon_t \equiv \Delta\ln\left(\frac{PPI_t}{P_t'}\right)$, where PPI_t is the PPI raw materials index and P_t' is the GDP deflator.

While the hypothesis that $\phi_1 + \phi_2 = 1$ cannot be rejected ($p-$value $= 0.98$), note that the coefficient $\widehat{\beta}^{OLS}$ is not statistically significant. In other words, the estimation of a standard hybrid New Keynesian Phillips curve over this time period suggests that, in fact, there *is* no Phillips curve. As the simulation results in Ashley and Verbrugge (2006a) suggest, however, a statistically insignificant β estimate does not necessarily imply the lack of a statistically significant Phillips curve relationship, since any frequency dependence in this relationship renders $\widehat{\beta}^{OLS}$ an inconsistent estimate.

Table 1 presents the coefficient estimates of interest for the analysis of frequency-dependence in the hybrid New Keynesian Phillips curve equation. Three specifications are considered:

- the "standard" hybrid New Keynesian Phillips curve (i.e., Equation (17) above, which ignores frequency dependence)

[24] Here and following, we quote results with robust (i.e., heteroskedasticity-consistent) standard error estimates, and without imposing the constraint that $\phi_2 = 1 - \phi_1$ (a constraint which the data does not reject in any of the specifications attempted).

- the "*a priori* calendar-based bands" model (which partitions the unemployment rate into bands with periods less than 6 quarters, between 6 and 12 quarters, and greater than 12 quarters),

and

- the "full partition" model (which partitions un_t into 13 frequency components).

Table 1 quotes both estimated t-statistics and estimated standard errors for the coefficient estimates (the results of robustness checks are in Appendix B). For the models which decompose un_t by frequency, we also report the p-value of the F-test with the null hypothesis $H_0 : \beta_1 = \beta_2 = ... = \beta_k$, for $k = 3$ (calendar-based bands) or $k = 13$ (full partition). A rejection of the null hypothesis indicates statistically significant frequency dependence.

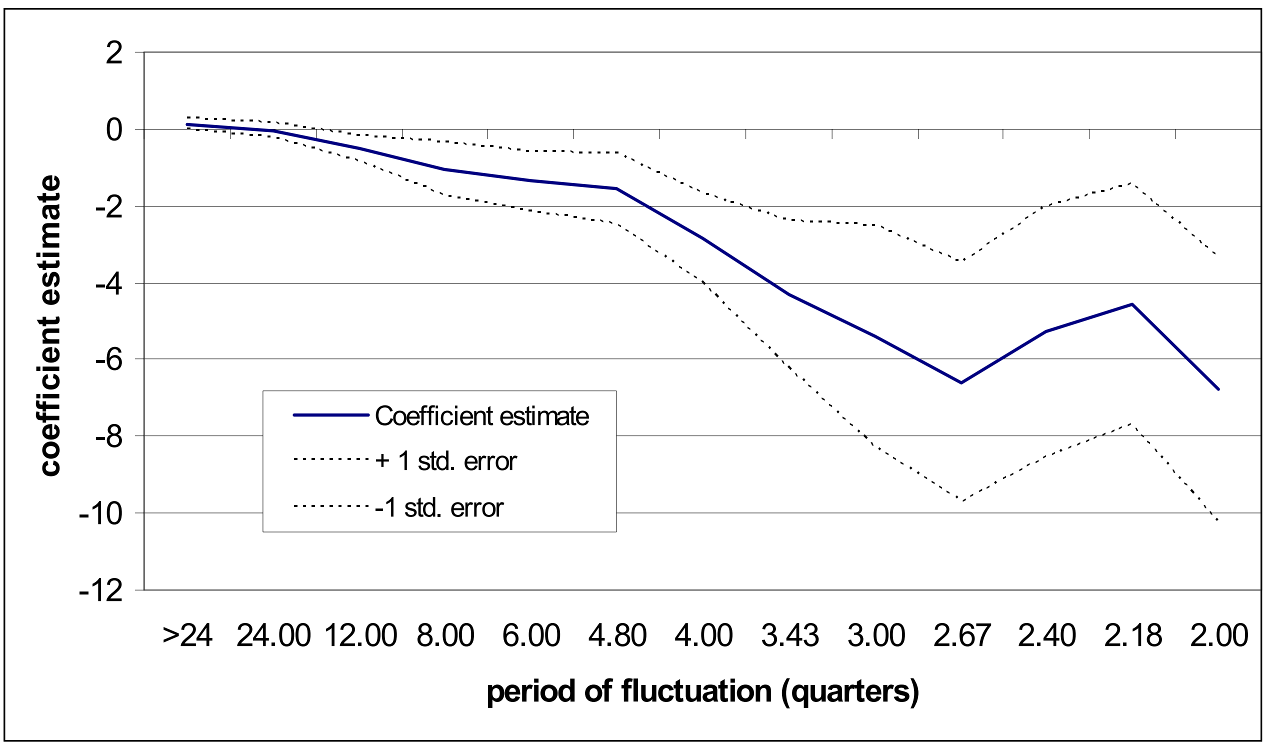
Table 1: Frequency Dependence in the Blanchard/Gali hybrid NKPC[29]

Coefficient	Classical	A priori calendar-based bands	Full partition (13 components)	*Period (quarters)*
ϕ_1	+ 0.32 ± 0.15 (+2.05)	+ 0.32 ± 0.15 (+2.15)	+ 0.52 ± 0.11 (+4.81)	
ϕ_2	+ 0.68 ± 0.23 (+3.01)	+ 0.71 ± 0.22 (+3.27)	+ 0.37 ± 0.13 (+2.91)	
F-test p-value (H_0: $\Sigma\phi = 1$)	0.98	0.80	0.19	
β	+ 0.02 ± 0.06 (+0.29)	+ 0.04 ± 0.09 (+0.44)	+ 0.14 ± 0.14 (+1.02)	≥ 24.00
			– 0.04 ± 0.18 (–0.24)	24.00
		– 0.62 ± 0.51 (–1.22)	– 0.51 ± 0.34 (–1.49)	12.00
			– 1.04 ± 0.69 (–1.51)	8.00
			– 1.34 ± 0.77 (–1.74)	6.00
		– 3.16 ± 1.71 (–1.85)	– 1.53 ± 0.92 (–1.67)	4.80
			– 2.86 ± 1.16 (–2.46)	4.00
			– 4.31 ± 1.91 (–2.26)	3.43
			– 5.40 ± 2.89 (–1.87)	3.00
			– 6.59 ± 3.11 (–2.11)	2.67
			– 5.28 ± 3.26 (–1.62)	2.40
			– 4.54 ± 3.13 (–1.45)	2.18
			– 6.79 ± 3.45 (–1.97)	2.00
F-test p-value (H_0: β_j equal)	—	0.07	0.015	

[29]OLS estimates are given ± robust standard error estimates, with estimated t ratios in parentheses. Note that the "full partition" estimates given in this table were smoothed in the same way as indicated immediately below this table for their display in Figure 2; the estimated standard errors quoted were appropriately modified to account for this smoothing. Robust (White) standard error estimates were used, as there is strong evidence of heteroscedasticity in these data. For the full partition: the p-value of the Jarque-Bera test of normality is 0.69. There is some evidence for ARCH(1) errors. Typically, some autocorrelation remains in the residuals. (See robustness checks in the appendix.)

While the results in Table 1 are informative for the "classical" and "calendar-based" results, a visual representation of the variation in $\widehat{\beta}_1 ... \widehat{\beta}_{13}$ is more readily interpretable. Figure 2 plots smoothed coefficient estimates; since $\widehat{\beta}_1 ... \widehat{\beta}_{13}$ vary considerably (due to sampling variation), these more clearly display the nature of the frequency dependence in the relationship. In particular, $\widehat{\beta}_j$ is replaced by a symmetric moving average, i.e. $\widehat{\beta}_j^s := \left(\frac{1}{4}\widehat{\beta}_{j-1} + \frac{1}{2}\widehat{\beta}_j + \frac{1}{4}\widehat{\beta}_{j+1} \right)$ for $j = 2, ..., 12$ (with $\widehat{\beta}_1^s = \frac{3}{4}\widehat{\beta}_1 + \frac{1}{4}\widehat{\beta}_2$ and $\widehat{\beta}_{13}^s = \frac{3}{4}\widehat{\beta}_{13} + \frac{1}{4}\widehat{\beta}_{12}$); the coefficient standard error estimates are adjusted accordingly.

Figure 2: Smoothed ("full partition") estimates of the impact of fluctuations in the unemployment rate upon inflation, decomposed by period of fluctuation



Five aspects of these results stand out. First, Table 1 reiterates the result (already noted in Section 2.2) that the coefficient on un_t in the "classical" model – i.e., ignoring frequency dependence – is not significantly different from zero. Second, note that using the "full partition" model (which flexibly allows for the possibility of frequency dependence in this coefficient) one can clearly reject the null hypothesis of no frequency dependence. In particular, the p-value for testing the null hypothesis that $\beta_1 = \beta_2 = ... = \beta_k$ is 0.015.[26] Once again, we note that the data suggests that there *is* a New Keynesian Phillips curve relation – but it applies only to unemployment fluctuations with periods less than about a year. Third, note that decomposing the unemployment rate using the *a priori* calendar-based bands yields only marginally statistically significant evidence for frequency dependence; the p-value is 0.07 for this test. This result is less sharp because the arbitrariness in the calendar-based bands restricts the model's ability to detect the actual pattern of frequency dependence present in the data. Fourth, while the estimated coefficients plotted in Figure 2 clearly vary, the results strongly indicate that fluctuations in the unemployment rate which last longer than 1 year have no statistically significant impact on inflation. In contrast, the null hypothesis that $\widehat{\beta}_1 ... \widehat{\beta}_{13}$ are all zero is rejected with a p-value less than 0.014, so there *is* a significant Phillips curve relationship once one allows for the frequency dependence in the relationship that is predicted by the Friedman-Phelps theory. To sum up, only relatively high-frequency fluctuations in the unemployment rate have an impact on inflation: the New Keynesian Phillips curve relationship is indeed inverse, but is restricted to relatively high frequencies.

As noted above, these findings are consistent with the Friedman-Phelps formulation: one might interpret transitory un_t fluctuations, i.e. those with periods less than around a year, as deviations from the natural rate (and thus negatively associated with contemporaneous inflation); whereas more persistent un_t fluctuations (with periods larger than this) might be interpreted as movements in the natural rate, with the implication that such persistent unemployment fluctuations are not associated with significant inflation co-movements. Note that these implicit fluctuations in the natural rate – i.e., all variation in un_t with periods longer than a year or so – are evidently fairly volatile (as in, e.g., King and Morley 2006), much more so than conventional estimates would suggest (see Williams 2004). Thus, under this interpretation of our results, a unit root process would be a rather poor approximation to the natural rate dynamics.

Finally, we find that the inflation impact of higher-frequency fluctuations in the unemployment rate is economically, as well as statistically, significant. To quantify and display the magnitude of

[26] This result is essentially unchanged under numerous alternative specifications; see Appendix B.

this impact, we constructed the time series $impact_t$:

$$impact_t := \left| \left(\sum_{j=1}^{13} \widehat{\beta}_j un_t^j \right) - \widehat{\beta}^{OLS} un_t \right|$$

This series quantifies the magnitude of the estimated impact of fluctuations in un_t on the inflation rate from allowing for frequency dependence in the relationship. Because the frequency dependence in the $\pi_t - un_t$ relationship is almost entirely at high frequencies, $impact_t$ is quite noisy. However, its mean value over the period 1984-2003 is 1.69, implying that these high-frequency fluctuations in the unemployment rate substantially altered the inflation rate to a significant extent over this time period. Evidently, an approximation to the relationship ingoring frequency dependence would have understated the impact of unemployment rate fluctuations on the inflation rate by well over one percentage point.

The New Keynesian Phillips curve literature has been perplexed both by the difficulty in locating evidence that π^e_{t+1} is an important determinant of inflation (see, e.g., Rudd and Whelan 2003), and by difficulty in finding a Phillips curve relationship at all in quarterly post-1984 data. Our results suggest that both of these difficulties are due to mis-specification of the form indicated above: failure to allow the coefficient β to vary across frequencies.

5 Conclusion

This paper's major contribution is empirical: we document that the absence, in post-1984 quarterly data, of a hybrid New Keynesian Phillips curve stems from mis-specification; in particular, we locate evidence for statistically-significant frequency dependence in the Blanchard/Gali (2005) hybrid New Keynsesian Phillips curve. Once this frequency-dependence is properly accounted for, we find that: a), this hybrid New Keynesian Phillips curve *does* exist in quarterly post-1984 data; and b), the detected pattern of frequency dependence is consistent with the Friedman-Phelps hypothesis of a (time-varying, fairly volatile) natural rate of unemployment. In particular, our results show that there is a significant inverse relationship for high-frequency fluctuations in the unemployment rate – roughly speaking, for fluctuations whose period is less than one year – and an insignificant relationship for more persistent unemployment fluctuations. A standard hypothesis test confirms that this pattern is statistically significant, at the 2% level.

What do these results mean? There are three main implications. First, our finding of statistically significant frequency dependence in this relationship suggests that previous New Keynesian Phillips curve coefficients are an admixture of several different frequency-specific coefficients, some negative and others zero. In particular, one implication of our results is that the apparent Phillips curve relationship can be expected to weaken or disappear in time periods when the unemployment rate fluctuates very smoothly.

Second, our results provide an empirical motivation for the development of New Keynesian theoretical models which can generate the kind of frequency dependence in the Phillips curve relationship which we find in the data.

Finally, our work poses challenges for forecasting and policy. If indeed only relatively high frequency fluctuations in the unemployment matter for inflation, this suggests that there may be a gain, in the context of forecasting, from using one-sided frequency decompositions. Furthermore, this also suggests that simple Taylor-type monetary policy rules – which form another of the key equations in most New Keynesian models – need to be re-formulated.

6 Appendix

6.1 Appendix A: Frequencies and periods associated with a 24-quarter rolling filtering window

The Table below indicates explicitly which frequencies (and periods, in months) will correspond to rows 2 and greater of the A matrix discussed in Section 3 with a rolling filtering window 24 quarters in length; row 1 corresponds to a within-window mean. A sinusoidal fluctuation in x_t with period equal to one of those listed here will appear entirely in the filtered series (D_t^j) containing that period; all other fluctuations will, to some degree, "leak" into the filtered series corresponding to adjacent frequency bands. Passband filters with a smaller degree of leakage can be formulated (e.g., Baxter and King, 1999), but do not yield filtered components which add up to the unfiltered series value.

allowed frequency	allowed period
0.042	24.00
0.083	12.00
0.125	8.00
0.167	6.00
0.208	4.80
0.250	4.00
0.292	3.43
0.333	3.00
0.375	2.67
0.417	2.40
0.458	2.18
0.500	2.00

6.2 Appendix B: Robustness checks

With respect to the two hypothesis tests $H_0 : \beta_1 = \beta_2 = ... = \beta_k$ and $H_0 : \beta_1 = ... = \beta_k = 0$, we obtained similar results:

- Imposing the restriction $\phi_1 + \phi_2 = 1$.
- Including additional lags of π. (In this case and in some of the following cases, this resulted in a statistically insignificant ϕ_2 estimate.)
- Approximating the 13 coefficient estimates with a Chebyshev polynomial; a 3rd-order polynomial was preferred by the data.
- Aggregating all fluctuations of the unemployment rate with period less than one year into one band (as long as lags of π were included).
- Aggregating all fluctuations of the unemployment rate with period less than one year into one band, and the remainder into another band.
- Running the regression over the period 1980:1-2003:12, rather than over 1984:1-2003:12. (This yielded p-values of 0.000 in the a priori case, and 0.001 in the full partition case. Furthermore, in this case including additional lags of π did nothing to diminish confidence in the statistical significance of ϕ_2.)
- Using a six-year window rather than a five-year window.
- Using a four-year window rather than a five-year window. (This yielded evidence in favor of frequency dependence – in both the *a priori* and full partition cases – only when additional lags of π were included.)

  However:

- Using the CPI or the PCE as the measure of inflation yielded results unsupportive of the existence of a hybrid NKPC.

6.3 Appendix C: Derivation of equation

The derivation of the relationship between real wage growth and the unemployment rate proceeds as follows.

$$\begin{aligned}
w_t &= c_t + \phi n_t^* + \xi_t \\
&= c_t + \phi n_t^* - \phi n_t + \xi_t + \phi n_t \\
&\equiv c_t + \phi n_t + \xi_t + \phi u_t \\
&\equiv mrs_t + \phi u_t
\end{aligned}$$

$$\begin{aligned}
w_t &= \gamma w_{t-1} + (1-\gamma)\, mrs_t \\
&\Rightarrow \frac{w_t - (1-\gamma)\, mrs_t}{\gamma} = w_{t-1} \\
&\Rightarrow w_t - w_{t-1} = \gamma \left(\frac{w_t - (1-\gamma)\, mrs_t}{\gamma} \right) + (1-\gamma)\, mrs_t - \left(\frac{w_t - (1-\gamma)\, mrs_t}{\gamma} \right) \\
\Delta w_t &= (\gamma - 1) \left(\frac{w_t - (1-\gamma)\, mrs_t}{\gamma} \right) + (1-\gamma)\, mrs_t \\
&= \frac{(\gamma-1)}{\gamma} w_t - \frac{(\gamma-1)(1-\gamma)\, mrs_t}{\gamma} + \frac{\gamma (1-\gamma)\, mrs_t}{\gamma} \\
&= \frac{(\gamma-1)}{\gamma} \left(w_t - (1-\gamma)\, mrs_t - \gamma mrs_t \right) \\
&= \frac{(\gamma-1)}{\gamma} \left(w_t - mrs_t \right)
\end{aligned}$$

Now substitute in the relation $w_t = mrs_t + \phi u_t$:

$$\begin{aligned}
\Delta w_t &= \frac{(\gamma-1)}{\gamma} \left(mrs_t + \phi u_t - mrs_t \right) \\
&= -\frac{(1-\gamma)\,\phi}{\gamma} u_t
\end{aligned}$$

6.4 Appendix D: An "Output-gap" hybrid NKPC

A standard "output-gap" hybrid NKPC is given by

$$\pi_t = \alpha + \phi_1 \pi_{t-1} + \phi_2 E_t \pi_{t+1} + \varphi x_t + \xi_t \tag{18}$$

where x_t is an output gap: $(y_t - y_t^*)$. If anything, researchers have had even more difficulty uncovering sensible and statistically-significant coefficient estimates using such specifications. As we note above, it is inappropriate to use output gap estimates which have been obtained using a two-sided filter, and probably undesirable to use output gap estimates which are the result of imposing *a priori* frequency selections. However, neither of these criticisms apply to using the methodology we use above; so it is of interest to determine whether or not mis-specification along the lines investigated in this paper is responsible for the miserable empirical performance of (18).

Using the same window as in Section 4 above, and using a forecast-combination approach as in Section 3.6 above, we partitioned log real gdp into its 13 frequency components, and estimated (18) as

$$\pi_t = \alpha + \phi_1 \pi_{t-1} + \phi_2 E_t \pi_{t+1} + \sum_{j=1}^{13} \varphi_j lrgdp_t^j + \xi_t \tag{19}$$

Results were only modestly supportive of a hybrid NKPC, as indicated in the table below; although the estimated pattern of frequency-dependence agrees with intuition, and there are a couple of statistically significant coefficient estimates, both of the F-test p-values are only significant at the 6% level.[27] (Note that large coefficient estimates do not, in themselves, signify much, since any particular frequency band may account for only a minimal fraction of total variance.) Evidently, the empirical failure of (18) does not result from a misapplication of two-sided filters in the estimation of y^*.

[27] Coefficients have been smoothed, as above. Additional lags of π_t are not statistically significant. The Chebyshev approximation did not support the existence of frequency-dependence in this relationship. The use capacity utilization as the estimate of x in (18) produced even weaker results: the F-test p-value for $H_0 : \varphi_i = \varphi_j \forall i \neq j$ was 0.065, the F-test p-value for $H_0 : \varphi_i \neq 0 \forall i$ was 0.086, and again the Chebyshev approximation yielded tests which were extremely unsupportive of the existence of a Phillips curve.

Mis-specification in Phillips Curve regressions

coefficient	estimate (*std. error*)
α	4.87 (4.44)
ϕ_1	0.29 (0.10)
ϕ_2	0.60 (0.11)
φ_1	$+0.72$ (2.84)
φ_2	1.01 (7.21)
φ_3	-4.27 (16.1)
φ_4	$+0.58$ (30.7)
φ_5	7.89 (52.4)
φ_6	-37.7 (79.3)
φ_7	-85.6 (96.6)
φ_8	-113 (90.2)
φ_9	-170 (106)
φ_{10}	-224 (102)
φ_{11}	-252 (83.8)
φ_{12}	-158 (85.9)
φ_{13}	$+15.6$ (121)
F-test p-value:φ_j *equal*	0.054
F-test p-value: $\varphi_j \neq 0$	0.056

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